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Asymmetric Attractive Particle Systems on Z: Hydrodynamic Limit for Monotone Initial Profiles

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We extend previous results on the preservation of local equilibrium for one-dimensional asymmetric attractive particle systems. The hydrodynamic behavior is studied for general monotone initial profiles.

KEY WORDS: Attractive systems; local equilibrium; hydrodynamic equation; entropy solutions; Euler scaling.

1. INTRODUCTION

The hydrodynamic behavior of interacting particle systems has been investigated by many authors. The situation is quite well understood for diffusive systems, but otherwise the results are still very partial. One important tool in this investigation has been *attractiveness*, a property shared by systems such as the simple exclusion process (SEP) and the zero-range process (ZRP). They both describe the evolution of infinitely many indistinguishable particles which jump on the sites of **Z** according to a translation-invariant probability $p(x, y) = p(y - x)$, following the exclusion rule (i.e., suppressing jumps on already occupied sites) for the SEP, and leaving a site with a rate which depends on the total number of particles at that site, for the ZRP. The extremal measures among those which are translation invariant, besides being invariant for one such process, form a one-parameter family of product measures, characterized by the particle

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density per site, and which we denote by $(\nu^a)_{0 \leqslant a < +\infty}$ [or $(\nu^a)_{0 \leqslant a \leqslant 1}$, for the SEP].[1,16,17]

In the nondiffusive case [e.g., when $p(\cdot)$ has a nonzero first moment], the hydrodynamic equation should be obtained under Euler scaling and it should be a (generally) nonlinear conservation law. This derivation, involving the proof of preservation of local equilibrium, has been obtained by several authors when the initial law is a product measure which agrees with some ν^a at the left of the origin and some ν^b at its right, i.e., when the initial density profile is a one-step function, or when the initial density profile is strictly monotone and will develop no shocks. We refer to refs. 3–6, 17, and 19 for these results obtained through different methods: in refs. 3, 4, 17, and 19 an explicit computation leads to the density profile; in refs. 5 and 6 the latter is obtained as the unique entropy solution to the hydrodynamic equation. In this paper, we combine these two approaches to extend the result to general monotonic initial conditions, as was conjectured in ref. 4, and announced in ref. 10.

The main tools of our proof are basic coupling, attractiveness of the processes, and the properties of entropy solutions to the hydrodynamic equations. First, we show how to reduce the problem to initial density profiles which are step functions and then we concentrate on such cases.

Since our main result (the conjecture 2.7 of ref. 4) concerns situations where shocks are developed, we must clarify that we do not say anything about the behavior at discontinuity points. Results about this exist in very particular cases[2,8,12,22]; they show loss of local equilibrium and also give the microscopic structure at the shock. Extensions to the multidimensional cases should be possible, and a first step has been obtained in refs. 14 and 15. For the particular case of nearest-neighbor asymmetric simple exclusion there are stronger results, based on the properties of second-class particles.[9]

2. PRELIMINARIES AND NOTATIONS

We study a class of Markov processes $(\eta_t)_{t \geqslant 0}$ called the "Misanthropes." They were introduced in ref. 7, and one such process is constructed on a suitable subset E of $X = \mathbf{N}^{\mathbf{Z}}$. We denote by $(S_t)_{t \geqslant 0}$ its semigroup, which is strongly continuous on the space $\mathscr{C}$ of bounded continuous functions on E. Its infinitesimal generator L applied to a bounded cylinder function f (i.e., f depends on finitely many coordinates) gives[7]

$$Lf(\eta) = \sum_{u, v \in \mathbf{Z}} b(\eta(u), \eta(v))\, p(u, v)[f(\eta^{u,v}) - f(\eta)] \tag{2.1}$$

where

$$\begin{aligned} \eta^{u,v}(x) &= \eta(u) - 1 && \text{if} \quad x = u \\ &= \eta(v) + 1 && \text{if} \quad x = v \\ &= \eta(x) && \text{if} \quad x \notin \{u, v\} \end{aligned}$$

provided $\eta(u) \geqslant 1$ and $u \neq v$; otherwise $\eta^{u,v} = \eta$. We make the following assumptions.

Assumptions 2.1.

(a) $b\colon \mathbf{N} \times \mathbf{N} \to [0, +\infty)$ is bounded, $b(0, \cdot) \equiv 0$, $b(1, j) > 0$ for all j, $b(\cdot, j)$ is increasing for each j, and $b(i, \cdot)$ is decreasing for each i.

(b) $b(i, j)/b(i+1, j-1) = b(i, 0)\, b(1, j)/b(i+1, 0)\, b(1, j-1)$ for all $i \geqslant 1$, $j \geqslant 1$; $b(i, j) - b(j, i) = b(i, 0) - b(j, 0)$ for all $i \geqslant 0$, $j \geqslant 0$.

(c) In this paper we also suppose $p(u, v) = p(v-u)$ for all v, u, where $p(u) \geqslant 0$, $\sum_u p(u) = 1$, $\sum_u |u|\, p(u) < +\infty$, and $\gamma \stackrel{\text{def}}{=} \sum_{u \in \mathbf{Z}} up(u) \in (0, +\infty)$. To avoid uninteresting complications we also assume that $p(\cdot, \cdot)$ is irreducible. Some examples we want to include are:

(i) Zero-range process (ZRP): $b(i, j) = g(i)$ with g bounded, nondecreasing, and such that $0 = g(0) < g(1)$, and $E = \mathbf{N}^{\mathbf{Z}}$.[1]

(ii) Simple exclusion process (SEP): $b(i, 0) = i \wedge 1$, $b(i, j) = 0$ if $j \geqslant 1$. In this case it is more natural to restrict the state space to $E = \{0, 1\}^{\mathbf{Z}}$.[17] Obviously Assumption 2.1a is *not* satisfied, but since the characterization of invariant measures is known, this case can be included. Thus we shall call Assumptions 2.1′ when besides the cases covered by Assumptions 2.1, we include the SEP, taking $E = \{0, 1\}^{\mathbf{Z}}$.

Remark 2.2. Under Assumptions 2.1 we can always take $E = \mathbf{N}^{\mathbf{Z}}$. Nevertheless if, for example, in the ZRP we substitute the assumption of g bounded by $\sup_k (g(k+1) - g(k)) < +\infty$, then we need to restrict the space E of allowed configurations, in order to construct a strongly continuous Markov semigroup. Doing this, as in ref. 1, our results still apply.

If $x \in \mathbf{Z}$ we denote by τ_x the shift operator acting on X by $\tau_x \eta(y) = \eta(y+x)$ for each $y \in \mathbf{Z}$; it acts on $\mathscr{C}$ by $\tau_x f(\eta) = f(\tau_x \eta)$, and on the space $\mathscr{P}$ of probability measures on X by $\int f\, d(\tau_x \mu) = \int (\tau_x f)\, d\mu$ for $\mu \in \mathscr{P}$, $f \in \mathscr{C}$. For $\mu \in \mathscr{P}$ we denote by $S_t \mu$ the law of η_t when η_0 is distributed according to μ, i.e., $\int f\, d(S_t \mu) = \int S_t f\, d\mu$. Let $\mathscr{I}(\mathscr{S})$ be the set of those $\mu \in \mathscr{P}$ that are $(S_t)_t$ $[(\tau_x)_x$ respectively$]$-invariant. Then, the extremal elements of $\mathscr{I} \cap \mathscr{S}$ form a one-parameter family of product measures $(\nu^a)_{a \in I}$. They are characterized by $a = \int \eta(0)\, \nu^a(d\eta)$ and

$$\frac{\nu^a(\eta(0)=n+1)}{\nu^a(\eta(0)=n)} = \frac{\nu^a(\eta(0)=1)}{\nu^a(\eta(0)=0)} \cdot \frac{b(1, n)}{b(n+1, 0)} \tag{2.2}$$

and under (a)–(c) above $I = [0, +\infty)$. Nevertheless, since for SEP we have restricted the state space to $\{0, 1\}^{\mathbf{Z}}$ we have $I = [0, 1]$ in this case.

Let us define

$$h(a) = \sum_{u \in \mathbf{Z}} up(0, u) \int b(\eta(0), \eta(u))\, d\nu^a(\eta)$$

which represents the flow of particles through any given site, under P_{ν^a}. (Here if $\mu \in \mathscr{P}$, P_μ denotes the law of $(\eta_t)_{t \geqslant 0}$ on the canonical space $D([0, +\infty), E)$ when η_0 is distributed according to μ.) It can be checked that h is continuous and we shall also assume:

Assumption 2.3. h is concave and continuously differentiable.

Recall that:

(i) For the ZRP, $h(a) = \gamma \int g(\eta(0))\, d\nu^a$, and

$$\begin{aligned} \nu^a(\eta(0)=k) &= \chi_\varphi^{-1} \varphi(a)^k / g(1) \cdots g(k) \qquad &&\text{if} \quad k \geqslant 1 \\ &= \chi_\varphi^{-1} &&\text{if} \quad k = 0 \end{aligned}$$

with $\varphi(a) = \int g(\eta(0))\, \nu^a(d\eta)$ and χ_φ is a normalizing constant. For example, if $g(k) = \mathbb{1}(k \geqslant 1)$, corresponding to a queuing system, we have $h(a) = \gamma a(1+a)^{-1}$, $0 \leqslant a < +\infty$.

(ii) For the SEP, $h(a) = \gamma a(1-a)$ with $0 \leqslant a \leqslant 1$.

For the construction and basic properties we refer to refs. 1, 7, 16, and 17.

We now recall the essential property of attractiveness (or monotonicity). The space X is endowed with the partial order $\eta \leqslant \xi$ if $\eta(u) \leqslant \xi(u)$ for all u. This induces the stochastic order on $\mathscr{P}$: $\mu_1 \leqslant \mu_2$ if there exists a probability measure $\bar{\mu}$ on $X \times X$ with $\bar{\mu}(A \times X) = \mu_1(A)$ and $\bar{\mu}(X \times A) = \mu_2(A)$ for all $A \in \mathscr{B}(X)$, and $\bar{\mu}\{(\eta, \xi): \eta \leqslant \xi\} = 1$.

The assumptions on $b(\cdot, \cdot)$ imply that if $\mu_1 \leqslant \mu_2$, then $S_t\mu_1 \leqslant S_t\mu_2$ for all $t \geqslant 0$. This monotonicity property is an essential tool for our proofs. Moreover, we shall use monotonicity when we employ the basic coupling, i.e., the construction on the same probability space of versions of the process, starting from several arbitrary configurations, in such a way that particles of the coupled processes evolve together as much as possible. Throughout this paper we denote with an overbar all that concerns a coupled process. Thus, if we couple (η_t) and (ξ_t) such that $\eta_0 \leqslant \xi_0$ we shall have a process $(\bar{\eta}_t, \bar{\xi}_t)$ with $\bar{\eta}_t \leqslant \bar{\xi}_t$ for all t, and we shall say that (ξ_t) is an upper bound for (η_t). See refs. 1, 7, and 17 for details on coupling techniques.

Note that $\nu^a \leqslant \nu^b$ for $a \leqslant b$.

Notation. The only convergence in $\mathscr{P}$ which we shall use is the w^*-convergence

$$\nu_n \xrightarrow{w^*} \nu \Leftrightarrow \int f\, d\nu_n \to \int f\, d\nu$$

for every bounded cylinder function f. If $x \in \mathbf{R}$, $[x]$ denotes the integer part of x.

3. MAIN RESULT

Our main result is the following:

Theorem 3.1. Let $\rho_0: \mathbf{R} \to I$ be increasing, bounded, and piecewise continuous, and for $\varepsilon > 0$ let μ_ε be the product measure such that

$$\mu_\varepsilon(\eta(k) = m) = \nu^{\rho_0(\varepsilon k)}(\eta(0) = m) \tag{3.1}$$

for $m \in \mathbb{N}$ ($m \in \{0, 1\}$ for the SEP) and $k \in \mathbf{Z}$. Let $\rho(\cdot, \cdot)$ be the entropy solution on $\mathbf{R} \times [0, +\infty)$ of

$$\begin{aligned} \partial_t \rho + \partial_x h(\rho) &= 0 \\ \rho(\cdot, 0) &= \rho_0(\cdot) \end{aligned} \tag{3.2}$$

Under Assumptions 2.1′ and 2.3 we have

$$\lim_{\varepsilon \to 0} \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon = \nu^{\rho(x,t)} \tag{3.3}$$

at all (x, t) continuity points of $\rho(\cdot, \cdot)$.

From here on we will make Assumptions 2.1′ and 2.3.

Remarks 3.2.

(i) Since h is concave, $\rho(\cdot, \cdot)$ is the unique generalized solution of Eq. (3.2) such that for every $t > 0$, $\rho(\cdot, t)$ has only increasing jumps. We refer to refs. 5, 10, 13, and 20 for a summary on the entropy condition.

(ii) As we mentioned in the introduction, the particular case of Theorem 3.1, when $\rho_0(x) = aI_{(-\infty,0)}(x) + bI_{[0,+\infty)}(x)$ with $0 < a < b$, has been proven in ref. 4. It has also been treated in ref. 6 with an independent and different method. But before these, the SEP was first studied in ref. 19 for $p(1)=1$, $a=1$, $b=0$; then in ref. 17 with $\gamma>0$, $a=1$, $b=0$. In ref. 5 the case $p(1)+p(-1)=1$ is treated; in refs. 3 and 22 the ZRP with $g(k)=\mathbb{1}(k\geqslant 1)$, $p(1)=1$, and any $a, b\geqslant 0$ was treated.

(iii) Ref. 3 treats also the case when $\rho_0(\cdot)$ is bounded, strictly decreasing, and of class C^1; this has been extended in ref. 4 to the models considered here, provided $h(\cdot)$ is strictly concave. The situation of decreasing ρ_0 is simpler, and local equilibrium is preserved everywhere. From what we shall do in this section the reader will easily verify that we may extend the above result to any $\rho_0(\cdot)$ bounded decreasing and piecewise continuous, when h is strictly concave. (See also Section 5.)

When $\rho_0(x)=aI_{(-\infty,0]}(x)+bI_{(0,+\infty)}(x)$ the measure μ_ε does not depend on ε, and we denote it by $\nu^{a,b}$. Let us now recall the result for this case, since we shall make intensive use of it in this section.

Theorem 3.3. The conclusion of Theorem 3.1 holds for one-step initial profiles, that is:

(a) If $a<b$ and h is concave

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\nu^{a,b}=\begin{cases}\nu^a & \text{if } \; x<v_c t\\ \nu^b & \text{if } \; x>v_c t\end{cases}$$

where $v_c=(h(b)-h(a))/(b-a)$.

(b) If $a>b$ and h is strictly concave

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\nu^{a,b}=\begin{cases}\nu^a & \text{if } \; x\leqslant th'(a)\\ \nu^{(h')^{-1}(x/t)} & \text{if } \; th'(a)\leqslant x\leqslant th'(b)\\ \nu^b & \text{if } \; x\geqslant th'(b)\end{cases}$$

This theorem has been proven by Andjel and Vares,[4] making essential use of the attractiveness. Another proof has been presented by Benassi and Fouque[5,6] [A.B. and J.P.F. take the present opportunity to say that there is an error in their use of the subadditive ergodic theorem (ref. 5, Proposition 1); they are working on its correction. Therefore we rely on ref. 4 for Theorem 3.3 above].

We are mainly concerned in extending part (a) of Theorem 3.3. Thus, we assume from now on that h is concave. To reduce the problem to initial profiles which are step functions we use some techniques in refs. 5 and 6, based on the properties of the entropy solution to Eq. 3.2. This is the content of Lemma 3.5 below. Before stating it we make the following.

Definition 3.4. We say that ρ_0 is a step function if there exist $n \geqslant 1$, $x_1 < \cdots < x_n$, $c_1, \ldots, c_n$ such that

$$\rho_0 = c_0 I_{(-\infty, x_1]} + \sum_{i=1}^{n-1} c_i I_{(x_i, x_{i+1}]} + c_n I_{(x_n, +\infty)}$$

If $c_i \neq c_{i+1}$ for all $i \in \{0, \ldots, n-1\}$, ρ_0 is said to be an n-step function.

Lemma 3.5. Assume (3.3) holds for any ρ_0 increasing step function. Then Theorem 3.1 follows.

Proof. Let us first remark that it suffices to consider ρ_0 as in Theorem 3.1 and for which we can find $R > 0$ such that $\rho_0(x) = \rho_0((-R \vee x) \wedge R)$ for all $x \in \mathbf{R}$. Indeed, given ρ_0 as in Theorem 3.1 and $R > 0$ we may take $\rho_{0,R}$, $\tilde{\rho}_{0,R}$ as in Theorem 3.1 and such that: (i) $\rho_{0,R} \leqslant \rho_0 \leqslant \tilde{\rho}_{0,R}$ everywhere; (ii) $\rho_{0,R}(x) = \rho_0(x) = \tilde{\rho}_{0,R}(x)$ for $|x| < R$; (iii) $\rho_{0,R}(x) = \rho_{0,R}((-R \vee x) \wedge R)$, $\tilde{\rho}_{0,R}(x) = \tilde{\rho}_{0,R}((-R \vee x) \wedge R)$ for all x.

Let $\mu_{\varepsilon,R}(\tilde{\mu}_{\varepsilon,R})$ be the measure defined as μ_ε but with ρ_0 replaced by $\rho_{0,R}$ ($\tilde{\rho}_{0,R}$ respectively). Then $\mu_{\varepsilon,R} \leqslant \mu_\varepsilon \leqslant \tilde{\mu}_{\varepsilon,R}$ and the attractiveness implies that for each (x, t)

$$\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_{\varepsilon,R} \leqslant \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon \leqslant \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \tilde{\mu}_{\varepsilon,R}$$

On the other hand, due to the hyperbolicity of Eq. (3.2) [$h'(\cdot)$ is bounded in any bounded interval], if $t > 0$ and $K \subseteq \mathbf{R}$ is a bounded interval, we can take $R > 0$ large enough so that $\rho_R(x, t) = \rho(x, t) = \tilde{\rho}_R(x, t)$ for $x \in K$, where ρ_R, ρ, and $\tilde{\rho}_R$ denote the entropy solution to Eq. (3.2) with initial conditions $\rho_{0,R}$, ρ_0, and $\tilde{\rho}_{0,R}$, respectively. (See, e.g., Theorem 1 of §3 in ref. 13 or the references therein for the one-dimensional case.)

Now, for ρ_0 satisfying the above extra condition, we can take two sequences of increasing step functions $(\alpha_n)_{n \geqslant 1}$, $(\beta_n)_{n \geqslant 1}$ such that

$$\alpha_n(x) \leqslant \rho_0(x) \leqslant \beta_n(x) \qquad \text{for all } x, \quad \text{all } n > 1 \tag{3.4a}$$

$$\alpha_n(x) \text{ increases in } n, \quad \beta_n(x) \text{ decreases in } n, \quad \text{for each } x \tag{3.4b}$$

$$\int_{\mathbf{R}} (\beta_n(x) - \alpha_n(x))\, dx \searrow 0 \qquad \text{as} \quad n \to +\infty \tag{3.4c}$$

Let $\alpha_n(x, t)$ ($\beta_n(x, t)$) be the entropy solution to Eq. (3.2) when $\alpha_n(\cdot, 0) = \alpha_n(\cdot)$ ($= \beta_n(\cdot)$, respectively), and let $\mu_{\varepsilon,n}(\tilde{\mu}_{\varepsilon,n})$ be the product measure such that $\mu_{\varepsilon,n}(\eta(x) = k) = v^{\alpha_n(\varepsilon x)}(\eta(0) = k)$ $(\tilde{\mu}_{\varepsilon,n}(\eta(x) = k) = v^{\beta_n(\varepsilon x)}(\eta(0) = k)$, respectively) for all $x \in \mathbf{Z}$, $k \in \mathbf{N}$. Thus, $\mu_{\varepsilon,n} \leqslant \mu_\varepsilon \leqslant \tilde{\mu}_{\varepsilon,n}$, and attractiveness implies

$$\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_{\varepsilon,n} \leqslant \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon \leqslant \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \tilde{\mu}_{\varepsilon,n} \tag{3.5}$$

for each $x \in \mathbf{R}$, $t \geqslant 0$, and $n \geqslant 1$.

According to monotonicity properties and *a priori* estimates for entropy solution[13] we also have

$$\alpha_n(x, t) \leqslant \rho(x, t) \leqslant \beta_n(x, t) \qquad \text{for all} \quad x, t \tag{3.6a}$$

where $\rho(x, t)$ is the entropy solution to Eq. (3.2), starting from ρ_0 and

$$\int_{\mathbf{R}} (\beta_n(x, t) - \alpha_n(x, t))\, dx \leqslant \int_{\mathbf{R}} (\beta_n(x) - \alpha_n(x))\, dx \tag{3.6b}$$

Moreover, h being concave, $\alpha_n(\cdot, t)$, $\rho(\cdot, t)$, and $\beta_n(\cdot, t)$ are increasing, for each $t > 0$.

From (3.6) we then get that if x is a continuity point of $\rho(\cdot, t)$ we must have

$$\lim_{n \to +\infty} \lim_{y \downarrow x} \beta_n(y, t) = \rho(x, t) = \lim_{n \to +\infty} \lim_{y \uparrow x} \alpha_n(y, t) \tag{3.7}$$

(3.5), (3.7), and the assumption of the lemma imply that (3.3) holds for ρ_0.

Remark 3.6. If ρ_0 were measurable, of bounded variation, and piecewise continuous, we could still approximate it by step functions satisfying (3.4). Equations (3.5) and (3.6) continue to hold, and since h is concave, entropy solutions to Eq. (3.2) have only positive jumps. We then easily obtain (3.7). Thus, the lemma can be extended to these functions, and our reason for stating it for increasing functions is simply that we do not have a proof of (3.3) for general step functions. (See Section 5.)

4. THE CASE OF ρ_0 INCREASING STEP FUNCTION

Proposition 4.1. Let $\rho_0 = a\mathbb{1}_{(-\infty, 0]} + c\mathbb{1}_{(0, \alpha]} + b\mathbb{1}_{(\alpha, +\infty)}$ with $0 \leqslant a < c < b$ ($\leqslant 1$ in the case of SEP). Assume $(h(c) - h(a))/(c - a) > (h(b) - h(c))/(b - c)$, and let x^*, t^*, x_1 be defined by

$$\alpha + t^* \frac{h(b) - h(c)}{b - c} = t^* \frac{h(c) - h(a)}{c - a} = x^* = x_1 + t^* \frac{h(b) - h(a)}{b - a}$$

Then:

(a) For $t < t^*$, (3.3) holds with

$$\rho(x, t) = a\mathbb{1}_{(-\infty, x(t))}(x) + c\mathbb{1}_{(x(t), y(t))}(x) + b\mathbb{1}_{(y(t), +\infty)}(x)$$

for $x \notin \{x(t), y(t)\}$, where

$$x(t) = t(h(c) - h(a))/(c - a), \qquad y(t) = \alpha + t(h(b) - h(c))/(b - c)$$

(b) For $t \geqslant t^*$, (3.3) holds with

$$\rho(x, t) = a\mathbb{1}_{(-\infty, z(t))}(x) + b\mathbb{1}_{(z(t), +\infty)}(x) \qquad \text{for} \quad x \neq z(t)$$

where

$$z(t) = x_1 + t(h(b) - h(a))/(b-a)$$

Remarks 4.2.

(a) Since h is concave, we always have $(h(c)-h(a))/(c-a) \geqslant (h(b)-h(c))/(b-c)$. The case of equality presents no difficulty: the monotonicity argument we shall use in the case of $t < t^*$ is enough to conclude that the same result applies to all $t > 0$.

(b) For the ZRP with $g(k) = \mathbb{1}(k \geqslant 1)$, t^* is defined by $\alpha + t^*\gamma(1+c)^{-1}(1+b)^{-1} = t^*\gamma(1+a)^{-1}(1+c)^{-1}$.

(c) For the SEP, t^* is defined by $\alpha + t^*\gamma(1-b-c) = t^*\gamma(1-a-c)$.

(d) Proposition 4.1b and Theorem 3.3 imply that if $t \geqslant t^*$ and $x \neq z(t)$, then

$$\lim_{\varepsilon \to 0} \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \tau_{-[x_1\varepsilon^{-1}]} \nu^{a,b} = \lim_{\varepsilon \to 0} \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon$$

for ρ_0 as above.

Let $\bar{\mu}_\varepsilon$ be a coupling of η with law μ_ε and $\tilde{\eta}$ with law $\tau_{-[x_1\varepsilon^{-1}]}\nu^{a,b}$ in such a way that

$$\begin{aligned} \bar{\mu}_\varepsilon\{(\eta, \tilde{\eta}) \colon \eta(x) = \tilde{\eta}(x) \quad &\text{for} \quad x \leqslant 0 \quad \text{or} \quad x > [\alpha\varepsilon^{-1}] \\ \tilde{\eta}(x) \leqslant \eta(x) \quad &\text{for} \quad 0 \leqslant x \leqslant [x_1\varepsilon^{-1}] \\ \eta(x) \leqslant \tilde{\eta}(x) \quad &\text{for} \quad [x_1\varepsilon^{-1}] < x \leqslant [\alpha\varepsilon^{-1}]\} = 1 \end{aligned} \tag{4.1}$$

This coupling satisfies the following "density balance relation":

$$\int \sum_{x=-\infty}^{[x_1\varepsilon^{-1}]} (\eta(x) - \tilde{\eta}(x))^+ \, d\bar{\mu}_\varepsilon = \int \sum_{x=[x_1\varepsilon^{-1}]+1}^{+\infty} (\tilde{\eta}(x) - \eta(x))^+ \, d\bar{\mu}_\varepsilon \tag{4.2}$$

The proof of Proposition 4.1 will be done in three steps: $t < t^*$, $t = t^*$, and finally $t > t^*$.

Proof of Proposition 4.1. Case $t < t^*$. We have $\nu^{a,c} \leqslant \mu_\varepsilon \leqslant \tau_{-[\alpha\varepsilon^{-1}]}\nu^{c,b}$ and $\tau_{-[\alpha\varepsilon^{-1}]}\nu^{a,b} \leqslant \mu_\varepsilon \leqslant \nu^{a,b}$. From this, the attractiveness of the process, and Theorem 3.3, for any fixed $t \in (0, t^*)$ we have:

(i) If $x(t) < x < y(t)$, then $\lim_{\varepsilon \to 0} \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon = \nu^c$.

(ii) If $x < t(h(b)-h(a))/(b-a)$, then $\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\mu_\varepsilon = \nu^a$.

(iii) If $x > \alpha + t(h(b)-h(a))/(b-a)$, then $\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\mu_\varepsilon = \nu^b$.

However, this first argument is not enough, since it does not give the hydrodynamic limit for $t(h(b)-h(a))/(b-a) < x < x(t)$ and for $y(t) < x < \alpha + t(h(b)-h(a))/(b-a)$. From the attractiveness we have that $\{\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\mu_\varepsilon\}_{x,t,\varepsilon}$ is ω^*-compact, and, moreover, the measures $\tau_{[x\varepsilon^{-1}]}S_{t\varepsilon^{-1}}\mu_\varepsilon$ are stochastically increasing in x. Thus, it suffices to control the density of particles (Lemma 4.3 below) to conclude the proof for $t < t^*$. (The details are the same as in Proposition 3.5 of ref. 4.)

Lemma 4.3. Let t^* be as in Proposition 4.1. If $0 < t < t^*$ we have:

(a) There exist x, y such that $y < x(t) < x < y(t)$, and

$$\lim_{\varepsilon\to 0}\int \varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z)\, d(S_{t\varepsilon^{-1}}\mu_\varepsilon) = (x - x(t))c + (x(t) - y)a$$

(b) There exist x, y such that $x(t) < x < y(t) < y$ and

$$\lim_{\varepsilon\to 0}\int \varepsilon \sum_{z=[x\varepsilon^{-1}]}^{[y\varepsilon^{-1}]} \eta(z)\, d(S_{t\varepsilon^{-1}}\mu_\varepsilon) = (y - y(t))b + (y(t) - x)c$$

Proof. We only prove (a), since (b) is analogous. There are three cases to be considered:

1. $0 < (h(b)-h(c))/(b-c) < (h(c)-h(a))/(c-a)$
2. $(h(b)-h(c))/(b-c) < 0 < (h(c)-h(a))/(c-a)$
3. $(h(b)-h(c))/(b-c) < (h(c)-h(a))/(c-a) < 0$

Let $\Delta = \{(x, s) : x(s) < x < y(s), 0 < s < t^*\}$.

Case 2 (this is the simplest case). Take $y < 0 \wedge t(h(b)-h(a))/(b-a)$ so that $\lim_{\varepsilon\to 0}\tau_{[y\varepsilon^{-1}]}S_{s\varepsilon^{-1}}\mu_\varepsilon = \nu^a$ for each $s \in [0, t]$. Notice that in this case $(x, t) \in \Delta$ implies $(x, s) \in \Delta$ for any $s \in (0, t]$. We fix x so that $(x, t) \in \Delta$. Then

$$\begin{aligned}\int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z)\right) d(S_{t\varepsilon^{-1}}\mu_\varepsilon) &= \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z)\right) d\mu_\varepsilon \\ &\quad + \int_0^t \left(\int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z)\right) d(S_{s\varepsilon^{-1}}\mu_\varepsilon)\right) ds \qquad (4.3)\end{aligned}$$

From (i) and (ii) above, (2.1), and Assumption 2.1c, we get

$$\lim_{\varepsilon \to 0} \int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{s\varepsilon^{-1}}\mu_\varepsilon) = h(a) - h(c)$$

for any $s \in (0, t]$. Using Assumption 2.1c and that $\nu^a \leqslant \mu_\varepsilon \leqslant \nu^b$, we may apply the dominated convergence theorem in Eq. (4.3) to conclude that

$$\begin{aligned}\lim_{\varepsilon \to 0} \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t\varepsilon^{-1}}\mu_\varepsilon) &= xc - ya + t(h(a) - h(c)) \\ &= (x - x(t))c + (x(t) - y)a\end{aligned}$$

which proves case 2.

Case 1. Let $t_1 < t_2 < \cdots$ be defined recursively by $x(t_1) = \alpha$, $x(t_{n+1}) = y(t_n)$ for $n \geqslant 1$, so that $t_n \uparrow t^*$ as $n \uparrow +\infty$. For $t \leqslant t_1$ the argument of case 2 can be applied, yielding that for $y < 0 \wedge t(h(b) - h(a))/(b-a)$ and $x(t) \leqslant x \leqslant \alpha$ we have

$$\lim_{\varepsilon \to 0} \int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{s\varepsilon^{-1}}\mu_\varepsilon) = h(a) - h(c)$$

for any $s \in (0, t)$. Using (2.1) and the dominated convergence theorem we conclude that for $t \leqslant t_1$

$$\lim_{\varepsilon \to 0} \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t\varepsilon^{-1}}\mu_\varepsilon) = (x - x(t))c + (x(t) - y)a$$

Now let us assume part (a) of the Lemma, for $t \leqslant t_n$. If $t_n < t \leqslant t_{n+1}$, we write

$$\begin{aligned}&\int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t\varepsilon^{-1}}\mu_\varepsilon) \\ &\quad = \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t_n\varepsilon^{-1}}\mu_\varepsilon) \\ &\qquad + \int_{t_n}^{t} \left[\int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{s\varepsilon^{-1}}\mu_\varepsilon) \right] ds\end{aligned}$$

Now, if we take y as before and x such that

$$t_{n+1}(h(c) - h(a))/(c-a) \leqslant x \leqslant \alpha + t_n(h(b) - h(a))/(b-a)$$

then $(x, s) \in \Delta$ for each $s \in (t_n, t_{n+1})$. This yields

$$\lim_{\varepsilon \to 0} \int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{s\varepsilon^{-1}} \mu_\varepsilon) = h(a) - h(c)$$

Just as before, this gives the proof of part (a) for $t \leqslant t_{n+1}$. By induction, we have the proof in case 1.

Case 3. It is analogous to case 1, so we omit it.

Proof of Proposition 4.1. *Case* $t = t^*$. Let us fix $x < x^*$. (The case $x > x^*$ is completely analogous.) Again, since $v^a \leqslant \tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon \leqslant v^b$, the same argument used in Proposition 3.5 of ref. 4 and based on attractiveness and compactness, tells us it is enough to prove that for any $y < x$

$$\lim_{\varepsilon \to 0} \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t^*\varepsilon^{-1}} \mu_\varepsilon) = (x - y) a \tag{4.4}$$

We may take $t < t^*$, so that $x < x(s)$ for any $s \in [t, t^*)$. Write the expression

$$\begin{aligned} &\int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t^*\varepsilon^{-1}} \mu_\varepsilon) \\ &\quad = \int \left(\varepsilon \sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t\varepsilon^{-1}} \mu_\varepsilon) \\ &\quad\quad + \int_t^{t^*} \left[\int L\left(\sum_{z=[y\varepsilon^{-1}]}^{[x\varepsilon^{-1}]} \eta(z) \right) d(S_{t^*\varepsilon^{-1}} \mu_\varepsilon) \right] ds \end{aligned} \tag{4.5}$$

We have $\lim_{\varepsilon \to 0} \tau_{[z\varepsilon^{-1}]} S_{s\varepsilon^{-1}} \mu_\varepsilon = v^a$ for any $z \leqslant x$ and $s \in (t, t^*)$. Thus, just as in the previous proof, when ε tends to zero the first term on the rhs of (4.5) tends to $(x - y)a$, and the second term tends to zero. This proves the case $t = t^*$.

We have just obtained the macroscopic behavior at $t = t^*$ and $x \neq x^*$. It is the same as if the starting measure were $\tau_{-[x_1\varepsilon^{-1}]} v^{a,b}$. In order to consider the case $t > t^*$, we shall need something stronger in this direction.

Proposition 4.4. Let $\bar{\mu}_\varepsilon$ be the coupling of μ_ε and $\tau_{-[x_1\varepsilon^{-1}]} v^{a,b}$ as defined by (4.1). Let $(\bar{S}_t)_{t \geqslant 0}$ be the semigroup associated to the so-called basic coupling of two versions of our process. If $x \neq x^*$, then any weak limit point $\bar{\mu}$ of $\tau_{[x\varepsilon^{-1}]} \bar{S}_{t^*\varepsilon^{-1}} \bar{\mu}_\varepsilon$ satisfies $\bar{\mu}\{(\eta, \tilde{\eta}) : \eta = \tilde{\eta}\} = 1$.

Notation. $\tau_x(\eta, \tilde{\eta})(z, w) = (\eta(z + x), \tilde{\eta}(w + x))$. $\tau_x \bar{\mu}$ is defined accordingly for $\bar{\mu}$ on $\mathbf{N}^{\mathbf{Z}} \times \mathbf{N}^{\mathbf{Z}}$. We say that $\bar{\mu}$ is translation invariant if $\bar{\mu} = \tau_1 \bar{\mu}$.

For the proof of this proposition we need the following.

Lemma 4.5. There exists a coupling of the three processes $(\eta_{1,t})_{t\geqslant 0}$, $(\eta_{2,t})_{t\geqslant 0}$, and $(\eta_{3,t})_{t\geqslant 0}$ with initial measures ν^a, $\nu^{a,b}$, and ν^b, respectively, and satisfying: for each $\beta>0$, $t\geqslant 0$, we can find integers M, N so that

$$P(\eta_{2,t}(x)=\eta_{1,t}(x) \text{ for all } x\leqslant M)\geqslant 1-\beta \tag{4.6a}$$

$$P(\eta_{2,t}(x)=\eta_{3,t}(x) \text{ for all } x\geqslant N)\geqslant 1-\beta \tag{4.6b}$$

Proof. We couple the initial configurations in a natural way: take η_1 distributed as ν^a; add ξ particles at sites $x\geqslant 0$ in such a way that $\eta_2 =^{\text{def}} \eta_1+\xi$ (coordinatewise) is distributed according to $\nu^{a,b}$; then add ζ particles at negative sites so that $\eta_3=\eta_2+\zeta$ has distribution ν^b. The coupled evolution is analogous to Lemma 3.3 of ref. 4: η_1 particles have priority to jump, then ξ particles, and finally ζ particles at lower priority, at any given site. This is done in such a way that $\eta_{1,t}$, $\eta_{2,t}=\eta_{1,t}+\xi_t$, and $\eta_{3,t}=\eta_{2,t}+\zeta_t$ have the right marginals. The generator of this coupled process acting on cylinder functions f gives

$$\begin{aligned}\bar{L}f(\eta_1,\xi,\zeta)=&\sum b(\eta_1(x),\eta_3(y))\,p(x,y)[f(\eta_1^{x,y},\xi,\zeta)-f(\eta_1,\xi,\zeta)]\\&+\sum\,[b(\eta_2(x),\eta_3(y))-b(\eta_1(x),\eta_3(y))]\\&\times p(x,y)[f(\eta_1,\xi^{x,y},\zeta)-f(\eta,\xi,\zeta)]\\&+\sum\,[b(\eta_3(x),\eta_3(y))-b(\eta_2(x),\eta_3(y))]\\&\times p(x,y)[f(\eta_1,\xi,\zeta^{x,y})-f(\eta,\xi,\zeta)]\end{aligned}$$

(where $\eta_2=\eta_1+\xi$, $\eta_3=\eta_2+\zeta$).

The probability on the lhs of (4.6a) can be bounded by $P(\xi_t(x)>0$ for some $x<M)$ and we can easily bound this because of the boundedness of $b(\cdot)$ and $\sum_{y\in\mathbf{Z}}|y|\,p(y)<+\infty$ using suitable independent random walks for the ξ particles, as in Lemma 3.3 of ref. 4. Similarly for (4.6b).

Proof of Proposition 4.4. Let us fix $x<x^*$ (the proof for $x>x^*$ is analogous). As in Lemma 1 of ref. 6, we shall prove that $\bar{\mu}$ is translation invariant and invariant under $(\bar{S}_t)_{t\geqslant 0}$ so that $\bar{\mu}(\{\eta\geqslant\tilde{\eta}\}\cup\{\tilde{\eta}\geqslant\eta\})=1$ (ref. 1, Section 5; ref. 7; and ref. 17, Chapter VIII).

Since $\tau_1\bar{\mu}$ is a weak limit point of $\tau_{[x\varepsilon^{-1}]}\bar{S}_{t^*\varepsilon^{-1}}\bar{\mu}_\varepsilon$, in order to show that $\bar{\mu}=\tau_1\bar{\mu}$, we first couple the process $(\eta_{1,t},\tilde{\eta}_{1,t})$, $(\eta_{2,t},\tilde{\eta}_{2,t})$ of initial distributions $\bar{\mu}_\varepsilon$ and $\tau_1\bar{\mu}_\varepsilon$, respectively. Using this process, we obtain a coupling of $\bar{\mu}$ and $\tau_1\bar{\mu}$, which we denote by $(\bar{\mu},\tau_1\bar{\mu})$. As usual we assume that the initial coupling distribution $\bar{\lambda}_\varepsilon$ satisfies a relation similar to (4.1).

By monotonicity, $(\bar{S}_{t\varepsilon^{-1}}\bar{\lambda}_\varepsilon)\{\eta_1 \leqslant \eta_2\} = (\bar{S}_{r\varepsilon^{-1}}\bar{\lambda}_\varepsilon)\{\tilde{\eta}_1 \leqslant \tilde{\eta}_2\} = 1$ for any t, ε. Then, taking weak limits,

$$(\bar{\mu}, \tau_1 \bar{\mu})\{\eta_1 \leqslant \eta_2\} = (\bar{\mu}, \tau_1 \bar{\mu})\{\tilde{\eta}_1 \leqslant \tilde{\eta}_2\} = 1$$

By Theorem 3.3 and the case $t = t^*$ of Proposition 4.1, we have that all marginals are ν^a. It then follows that $(\bar{\mu}, \tau_1\bar{\mu})\{\eta_1 = \eta_2\} = (\bar{\mu}, \tau_1\bar{\mu})\{\tilde{\eta}_1 = \tilde{\eta}_2\} = 1$, and so $\bar{\mu} = \tau_1 \bar{\mu}$.

To show that $\bar{\mu} = \bar{S}_t \bar{\mu}$ for each $t > 0$, we consider the process $(\eta_{2,s}, \tilde{\eta}_{2,s})_{s \geqslant 0}$ with $\bar{S}_t \bar{\mu}_\varepsilon$ as initial distribution, since $\bar{S}_t \bar{\mu}$ is a weak limit of $\tau_{[x\varepsilon^{-1}]}\bar{S}_{t^*\varepsilon^{-1}}\bar{S}_t\bar{\mu}_\varepsilon$. Using Lemma 4.5 and the attractiveness, for any given $\beta > 0$ we may couple $(\eta_{2,s}, \tilde{\eta}_{2,s})$ with $(\eta_{3,s}, \tilde{\eta}_{3,s})$ and $(\eta_{4,s}, \tilde{\eta}_{4,s})$ with initial laws $\tau_M \bar{\mu}_\varepsilon$ and $\tau_{-M}\bar{\mu}_\varepsilon$ for suitable $M \geqslant 1$ in such a way that the initial coupling measure $\bar{\lambda}_\varepsilon$ satisfies

$$\bar{\lambda}_\varepsilon\{\eta_3 \leqslant \eta_2 \leqslant \eta_4\} \wedge \bar{\lambda}_\varepsilon\{\tilde{\eta}_3 \leqslant \tilde{\eta}_2 \leqslant \tilde{\eta}_4\} \geqslant 1 - \beta$$

Using the translation invariance of $\bar{\mu}$ and arguing as in its proof, we get $\bar{\mu} = \bar{S}_t \bar{\mu}$.

Thus, we know that: (i) $\bar{\mu}\{(\eta, \tilde{\eta}) : \eta \geqslant \tilde{\eta} \text{ or } \tilde{\eta} \geqslant \eta\} = 1$; (ii) $\bar{\mu}$ is translation invariant; (iii) both marginals of $\bar{\mu}$ are ν^a so that $\lim_{n \to +\infty} 1/n \sum_{i=1}^{n} (\eta(i) - \tilde{\eta}(i)) = 0$ $\bar{\mu}$-a.s.; (iv) $\{\eta \geqslant \tilde{\eta}\}$ and $\{\tilde{\eta} \geqslant \eta\}$ are both translation invariant. From this one gets that $\bar{\mu}\{\eta = \tilde{\eta}\} = 1$ (this argument is due to P. Ferrari, whom we thank for the discussion, and appears in Proposition 2.19 of Ref. 9).

Proof of Proposition 4.1. *Case* $t > t^*$. Let us fix $t > t^*$. By Proposition 4.4, for any $x \neq x^*$

$$\lim_{\varepsilon \to 0} \tau_{[x\varepsilon^{-1}]} \bar{S}_{t^*\varepsilon^{-1}} \bar{\mu}_\varepsilon\{\eta(0) \neq \tilde{\eta}(0)\} = 0$$

From this we easily see that for any $-\infty < z < l < +\infty$ and any $\delta > 0$

$$\lim_{\varepsilon \to 0} \bar{S}_{t^*\varepsilon^{-1}} \bar{\mu}_\varepsilon \left\{ \varepsilon \sum_{y = [z\varepsilon^{-1}]}^{[l\varepsilon^{-1}]} \mathbb{1}_{(\eta(y) \neq \tilde{\eta}(y))} > \delta \right\} = 0 \tag{4.7}$$

We first claim that (4.7) is still true when we change t^* by t. Indeed, suppose this were not the case, so that we could take $\delta > 0$ and $z < l$ such that

$$\overline{\lim_{\varepsilon \to 0}}\, \bar{S}_{t\varepsilon^{-1}} \bar{\mu}_\varepsilon \left\{ \varepsilon \sum_{y = [z\varepsilon^{-1}]}^{[l\varepsilon^{-1}]} \mathbb{1}_{(\eta(y) \neq \tilde{\eta}(y))} > \delta \right\} > \delta$$

Since $\sum_y |y|\, p(y) < +\infty$, a simple comparison with independent particles allows us to find $k < +\infty$ so that for any $1 > \varepsilon > 0$, with probability at least $1-\delta/2$ no particle at the left of $[(z-k)\,\varepsilon^{-1}]$ (right of $[(l+k)\,\varepsilon^{-1}]$) at time $t^*\varepsilon^{-1}$ went to the right of $[z\varepsilon^{-1}]$ (left of $[l\varepsilon^{-1}]$ respectively) up to time $t\varepsilon^{-1}$. But the coupled dynamics $\bar{S}_t$ cannot create discrepancies between $\eta(\cdot)$ and $\tilde{\eta}(\cdot)$ and so we would have

$$\varlimsup_{\varepsilon \to 0} \bar{S}_{t^*\varepsilon^{-1}} \bar{\mu}_\varepsilon \left\{ \varepsilon \sum_{y=[(z-k)\,\varepsilon^{-1}]}^{[(l+k)\,\varepsilon^{-1}]} \mathbb{1}_{(\eta(y) \neq \tilde{\eta}(y))} > \delta \right\} > \delta/2$$

contradicting (4.7), and proving the claim.

By Theorem 3.3, if $z(t) < z < l$, then $\varepsilon \sum_{y=[z\varepsilon^{-1}]}^{[l\varepsilon^{-1}]} \tilde{\eta}(y)$ converges in probability to $(l-z)b$. Combining this with the relation $\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon \leqslant \nu^b$ and (4.7), with t in the place of t^* we get that $\tau_{[z\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon \to \nu^b$ if $z > z(t)$. Similarly, this limit is ν^a if $z < z(t)$. This completes the proof of Proposition 4.1.

We need now to extend the result for any number of steps in the initial profile. This is the content of the next result.

Proposition 4.6. Theorem 3.1 holds when ρ_0 is an increasing step function.

Proof. We already know this for ρ_0 a 1- or 2-step function. Let us assume the result is proven for ρ_0 increasing and a k-step function, with $k \leqslant n$. Now let us assume ρ_0 is an increasing and $(n+1)$-step function. The arguments used for $t \leqslant t^*$ in Proposition 4.1 give us the result up to the time $\tilde{t}$ of the first collision between discontinuity lines of $\rho(\cdot,\cdot)$. At this point it is easy to obtain an analogue of Proposition 4.4 by suitably coupling μ_ε and a measure $\tilde{\mu}_\varepsilon$ whose profile is a k-step function, for $k \leqslant n$. After this we can easily conclude the proof, as in Proposition 4.1.

5. CONCLUDING REMARKS

Concerning decreasing initial profiles, the situation is much simpler, in agreement with the fact that the hydrodynamic equation will develop no shocks. When ρ_0 is strictly decreasing and continuous the preservation of local equilibrium has been proven in ref. 4 (Theorem 2.10). The technique used here (approximation by step initial profiles) easily yields the general decreasing case, and we can state the following result.

Theorem 5.1. Let μ_ε be as in Theorem 3.1, where ρ_0 is now decreasing, bounded, and piecewise continuous. Under Assumptions 2.1′

and 2.3, and requiring h to be strictly concave we have that (3.3) holds, where now $\rho(x, t)$ is the unique classical solution of Eq. (3.2).

Proof. For ρ_0 any decreasing step function, the proof is very simple and based on the same arguments as in the case $t < t^*$ of Proposition 4.1. Indeed, if ρ_0 is as in Proposition 4.1, but now $0 \leqslant b < c < a$, we can simply use $\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \nu^{a,c}$ and $\tau_{[x\varepsilon^{-1}] - [\alpha\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \nu^{a,b}$ as upper bounds for $\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon$. Applying Theorem 3.3b, we get $\nu^{\rho(x,t)}$ as upper bound for any possible limit point of $\tau_{[x\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu_\varepsilon$ $(\varepsilon \downarrow 0)$. Similarly for the lower bound. The reasoning for the induction step is just the same. This, together with Lemma 3.5, where α_n, β_n can be taken decreasing if ρ_0 is so, allows us to treat the case of decreasing initial profiles.

One natural question concerns the extension of Theorem 3.1 to ρ_0 not necessarily monotonic. We do not have a complete proof of such a result, but only partial answers for the ZRP and the SEP.[11,15]

If one is not asking about local equilibrium, but only convergence of the density field, general bounded, measurable initial profiles can be allowed. This has been recently studied in ref. 18.

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